Studies on Invariants of Analytic Singularities
Research Report

Adam Parusiński

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During my stay in Japan, from April 28 till May 27 2013, besides everyday discussions with the host Professor Satoshi Koike of Hyogo Univeristy of Teacher Education, I have been discussing mathematics with many Japanese mathematicians including Professor Shizo Izumi from Kinki University and Dr. Masaru Hasegawa from Saitama University. During my stay in Saitama, May 7- May 12, I have been working with Professor Toshizumi Fukui on the inverse theorem for arc-analytic morphisms, see below. I have also discussions with students of Professor Satoshi Koike.

This project is focused on invariants of real analytic functions germs related to two important equisingularity relations: blow-analytic equivalence and bi-lipschitz equivalence.

Blow-analytic equivalence.
The classification of real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ requires a good natural equivalence relation. Blow-analytic equivalence was proposed by T.-C. Kuo at the beginning of the 80's. Roughly speaking two function germs are blow-analytically equivalent if they have (analytically) isomorphic resolution spaces. There is a complete classification of blow-analytic types if $n = 2$ given in [S. Koike and A. Parusiński, *Blow-analytic equivalence of two variable real analytic function germs*, J. of Algebraic Geometry **19** (2010), 439–472.] and only partial results for $n > 2$. The latter are based on "motivic type invariants" introduced in [S. Koike and A. Parusiński, *Motivic-type invariants of blow-analytic equivalence* Ann. Inst. Fourier **53** (2003), 2061–2104] and In [38] and then developed by G. Fichou in the blow-Nash (algebraic) case.

Another open and important problem is the relation of blow-analytic equivalence and its invariant to more geometric nature of singularities as for instance bi-lipschitz or C^k-equivalence or the asymptotic behavior of curvature. Again the relation to bi-lipschitz or C^k-equivalence is well understood only in two dimensional case [S. Koike and A. Parusiński, *Equivalence relations for two variable real analytic function germs,* J. Math. Soc. Japan Volume 65, Number 1 (2013), 237-276].

Bi-Lipschitz and Metric Geometry.
Bi-lipschitz geometry of singularities is a very active area. The Lipschitz stratification, that refines the Whitney one, was first introduced by T. Mostowski in the complex analytic set-up and then devolped in real subanalytic set-up by A. Parusiński. Bi-Lipschitz equivalence of real analytic function germs has been studied by J.-P. Henry and A. Parusiński, and then by M. Ruas and A. Fernandes. There are several recent papers by L. Birbrair, W. Neumann, and A. Pichon on the complex normal singularity case.

The asymptotic behavoiur of curvature has been studied in the complex case by R. Langevin, L. Ness, and J'.Fu. By constructing the normal cycle to a singular set, J.Fu extended this work the real subanalytic case. Our investigations were motivated by recent papers of Koike, Kuo and Paunescu on the (non)concentration of curvature for analytic function germs of two real variables that can be partially but not completely studied by the real tree model of S. Koike and A. Parusiński (the one that serves to classify blow-analytic equivalence classes).

Lectures :

1. April 30, 2013, 15h00 - 16h00, Hyogo University of Teacher Education.
   Colloquium: *Introduction to Real Algebraic Geometry Part I: Geometry of semi-algebraic sets*

   Abstract: *We present an introduction to semi-algebraic sets. Semi-algebraic sets are subsets of affine spaces over real numbers defined by finitely many polynomial equations and singularities. Seidenberg-Tarski theorem says that the image of a semi-algebraic set by a polynomial mapping is again semi-algebraic. We discuss the basic properties of semi-algebraic sets such as: the curve selection lemma, stratifications and cellular decompositions.*

2. May 9, 2013, 16200 - 17h20, Saitama University.
   Lecture: *Normal cycle of semi-algebraic sets.*

   Abstract: *We give an introduction to the normal cycle of a semi-algebraic set. The normal cycle is a geometric object, more precisely a*

Lagrangian variety, that reflects such geometric features as curvatures measures, vanishing topology and stratified Morse Theory. It provides a geometric interoperation of the Stiefel-Whitney characteristic classes of singular semi-algebraic sets.

3. May 21, 2013, 15h00 - 16h00, Hyogo University of Teacher Education. Colloquium: *Introduction to Real Algebraic Geometry Part I: Normal cycle of semi-algebraic sets*

First results :
Together with Professor Toshizumi Fukui we have obtained the following Inverse Function Theorem for arc-analytic homeomorphism that was stated in our earlier paper (joint with Krzysztof Kurdyka) as an open problem.

Theorem [(Inverse Function Theorem for Arc-Analytic Homeomorphisms)] Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semialgebraic homeomorphism. If h is arc-analytic and $|\det(dh)| \geq c$ for a constant $c > 0$, then h is an arc-analytic isomorphism (i.e. h^{-1} is arc-analytic and $\det(dh)| \leq C$).

This general theorem gives the following important corollaries.

Corollary 1
Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semialgebraic homeomorphism such that h and h^{-1} are blow-analytic. If h is lipschitz then so is h^{-1}.

Corollary 2
Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semialgebraic homeomorphism such that h and h^{-1} are blow-analytic. If h is C^k, $k = 1, 2, \ldots, \infty, \omega$, then so is h^{-1}.

To show this theorem we need a new version of Kontsevich's change of variable formula that holds for the maps that are not necessarily birational. The proof uses as essential ingredients: the proof of real analytic version of the classical change of variable formula [S. Koike and A. Parusiński, *Motivic-type invariants of blow-analytic equivalence*, Ann. Inst. Fourier **53** (2003), 2061–2104] and the virtual Betti numbers, the additive invariants of real algebraic sets, developed in the joint papers of C. McCrory and A. Parusiński.